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One-dimensional closure models for three-dimensional incompressible viscoelastic free jets: von Kármán flow geometry and elliptical cross-section

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In this paper we derive one-space-dimensional, reduced systems of equations (one-dimensional closure models) for viscoelastic free jets. We begin with the three-dimensional system of conservation laws and a Maxwell–Jeffreys constitutive law for an incompressible viscoelastic fluid. First, we exhibit exact truncations to a finite, closed system of one-dimensional equations based on classical velocity assumptions of von Kármán (1921). Next, we demonstrate that the three-dimensional free-surface boundary conditions overconstrain these truncated systems, so that only a very limited class of solutions exist. We then proceed to derive approximate one-dimensional closure theories through a slender-jet asymptotic scaling, combined with appropriate definitions of velocity, pressure and stress unknowns. Our non-axisymmetric one-dimensional slender-jet models incorporate the physical effects of inertia, viscoelasticity (viscosity, relaxation and retardation), gravity, surface tension, and properties of the ambient fluid, and include shear stresses and time dependence. Previous special one-dimensional slender-jet models correspond to the lowest-order equations in the present asymptotic theory by an *a posteriori* suppression to leading order of some of these effects, and a reduction to axisymmetry. We thereby: (i) derive existing one-dimensional models from the three-dimensional free surface boundary-value problem; (ii) clarify the sense of the one-dimensional approximation; (iii) deduce new one-dimensional closure models for non-axisymmetric viscoelastic free jets.

1. Introduction and history

There are at least two motivations for one-space-dimensional models of free, three-dimensional fluid jets. For engineering applications such as ink-jet printing, polymer extrusion and fibre spinning, there is a need to reproduce and predict experimental jet phenomena with a simple and tractable system of equations. This has been a dominant theme in the history of the subject. Secondly, in light of the measured success of one-dimensional models in certain specific jet applications, it is natural to ask why the lower-dimensional models are able to model three-dimensional phenomena. Can these one-dimensional models be derived in some approximate sense from the full three-dimensional free boundary-value problem (b.v.p.)?

Our primary purpose in this paper is to answer this question. We will: (i) derive one-dimensional models for free jets from the three-dimensional free surface b.v.p.; and (ii) clarify the sense in which the one-dimensional models approximate and are consistent with the full three-dimensional b.v.p. The analytic framework is based on averaging over two-dimensional jet cross-sections and an asymptotic scaling in a slenderness ratio. In answering this question, we find that existing one-dimensional models correspond to particular specifications of fluid and flow properties within one comprehensive theory.

The one-dimensional jet models are a truncation of the full three-dimensional system (which has infinite modes in three space and one time dimensions) to a finite number of unknown (modes) in one space (axial coordinate) and one time dimensions.

Analogous truncations occur in all numerical simulations of three-dimensional fluids. For example, in spectral methods one chooses to truncate at some finite term in the Fourier mode expansion. In specific applications, often one exploits special properties and/or symmetries of the full three-dimensional b.v.p. to truncate modes and/or spatial dimensions. Two examples are the exact truncation to vortex-sheet and vortex-layer equations for three-dimensions Euler flows, and the von Kármán (1921) velocity profile assumption for three-dimensional Newtonian flow between rotating concentric cylinders.

When the truncation scheme is not an exact reduction of the full system, an art arises as to the best way to 'close the system' and produce the same number of equations as unknowns (a closure model). Rarely can or does one qualify the sense in which a truncated, non-exact closure model approximates the full system. The proof is usually by comparison with experiments. A novelty of the present application to three-dimensional jet flows is that we deduce asymptotically valid, one-dimensional closure models from the full three-dimensional b.v.p. The asymptotics is based on a slender-jet geometry.

We reiterate that the focus of this paper is the derivation of consistent one-dimensional closure models for free viscoelastic jets from the full three-dimensional b.v.p. The solutions and consequences of these models in applications to free-jet phenomena are given in Bechtel, Forest & Lin (1987, 1988*a*), Bechtel, Forest & Hyman (1988) and Bechtel, Bogy & Forest (1986). A variety of future applications are planned.

Throughout this paper we refer to the unknowns as *modal variables*, by analogy with amplitude variables in Fourier mode expansions. We then refer to the reduced equations that govern these unknowns as *modal equations*.

The remainder of this paper is organized as follows. In §2 we exhibit exact truncations, for inviscid, Newtonian, and non-Newtonian *unbounded* flows, to a finite, closed system of one-dimensional modal equations. These results follow from a 'separation of variables' structure of the Navier–Stokes equations which von Kármán (1921) applied to study axisymmetric flow between rotating concentric cylinders. More recently, Phan-Thien (1983) extended this structure to axisymmetric flows of an upper convected Maxwell fluid. Here we note the extension of this exact finite mode truncation to elliptical symmetry and a more general (Maxwell–Jeffreys) constitutive model. (These exact truncations will arise later as the 'zeroth-order' basis of our perturbation theory for bounded, free-surface flows.)

In §3, we investigate the constraints imposed on the exact one-dimensional closure models of §2 when a free surface is introduced. We demonstrate that the three-dimensional interfacial boundary conditions overconstrain the previously closed system of equations, so that only trivial solutions exist. We then proceed to show that an approximate one-dimensional closure theory can be salvaged in an appropriate scaling limit. In essence, we exploit the exact one-dimensional closure models of §2 in a perturbation expansion, with a slenderness ratio as the perturbation parameter.

There is a long history (Matovich & Pearson 1969; Kase 1974; Denn, Petrie & Avenas 1975; Fisher & Denn 1976; Denn & Marrucci 1977; Schultz & Davis 1982; Denn 1983; Joseph, Matta & Nguyen 1983; Tanner 1985; Phan-Thien & Caswell 1986; Gupta, Puszynski & Sridhar 1986) of approximate one-dimensional models for free Newtonian and viscoelastic jets, often referred to as the 'thin filament' or 'slenderness' approximation, or 'nearly elongational' flows. The original formulation is due to Matovich & Pearson (1969) in the study of fibre spinning. Many authors have since adopted their perturbation scheme, which is purely formal since the perturbation parameter is not identified in terms of any specific dimensionless flow or fluid parameter. This heuristic aspect of the theory clouds applications of the scheme since there is no physical scaling hypothesis. In general, the range of assumptions and validity of individual one-dimensional jet models is unknown. The need for an analysis which catalogues the full range of assumptions, and indicates when the one-dimensional approximation breaks down, is expressed in Petrie (1979), Denn (1983) and Tanner (1985). Also the existing models are presented and applied under a variety of *a priori* restrictions (e.g. in the absence of one or more of time dependence, shear stresses, gravity, inertial effects and surface tension), as dictated by the particular applications. All existing models are axisymmetric.

(A second group of one-dimensional jet models are based on posited self-consistent one-dimensional models (cf. Naghdi 1979, 1981; Green, Naghdi & Wenner 1974*b*; Antmann 1972; Bechtel *et al.* 1986). The connections between the posited one-dimensional models and derivations from the three-dimensional free-surface boundary-value problem are discussed in Bechtel *et al.* 1987).

In §§4–6 we derive an asymptotically valid, one-dimensional theory of slender-jet closure models. The theory is comprehensive, in that we begin with the full three-dimensional free-surface boundary-value problem, with the following physical effects incorporated: time dependence, shear stresses, inertial effects, viscoelasticity (viscosity, relaxation and retardation effects), gravity, surface tension and properties of the ambient fluid as they appear in the free-surface interfacial conditions. In this way we develop the general context under which every one-dimensional jet closure model (with these physical effects, constitutive law and choice of modal variables) is deduced.

As is shown in §7, existing one-dimensional jet theories correspond in this general framework to the lowest-order equations in the asymptotic expansion, with *a posteriori* suppression to leading order of many of the physical effects. We thereby derive previous one-dimensional models from the three-dimensional free-surface boundary-value problem and clarify the sense of the one-dimensional closure model approximation.

In addition, we have deduced new, asymptotically valid, one-dimensional closure models for viscoelastic free jets. One particular new feature is the extension to elliptical free-surface symmetry (this generalization was suggested by the posited one-dimensional models of Caulk & Naghdi (1979*a*, *b*) for inviscid and Newtonian elliptical jets). Moreover, *higher-order corrections* are available from this analytic framework, both from within a specific model and due to physical effects that are suppressed in the lowest-order equations.

We begin with the equations of motion for an arbitrary, incompressible three-dimensional continuum:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right)=\rho\boldsymbol{g}+\operatorname{div}\boldsymbol{T}, \qquad (1.1a)$$

$$\boldsymbol{T}=-p\boldsymbol{I}+\hat{\boldsymbol{T}}=\boldsymbol{T}^{\mathrm{T}}, \qquad (1.1b)$$

$$\operatorname{div}\boldsymbol{v}=0. \qquad (1.1c)$$

Here $\boldsymbol{v}$ is the velocity, $\hat{\boldsymbol{T}}$ is the determinate part of the stress tensor $\boldsymbol{T}$, p is the constraint pressure, ρ is the mass density (assumed constant), and $\rho\boldsymbol{g}$ is the gravitational body force. Equations (1.1a) and (1.1b) are balance laws for linear momentum and angular momentum, and (1.1c) is the incompressibility constraint.

For a three-dimensional continuum, a constitutive law must be adjoined to determine the unknown stress $\hat{\boldsymbol{T}}$. In this paper we consider viscoelastic fluids and adopt a Maxwell–Jeffreys constitutive model:

$$\hat{\boldsymbol{T}}+\lambda_1\frac{\mathrm{D}}{\mathrm{D}t}\hat{\boldsymbol{T}}=2\eta\left(\boldsymbol{D}+\lambda_2\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{D}\right), \qquad (1.2a)$$

where $\boldsymbol{D}$ and the 'rate' $\mathrm{D}/\mathrm{D}t$ are given below. The operator $\mathrm{D}/\mathrm{D}t$ must be suitably invariant; we choose a one-parameter family with rate parameter a,

$$\frac{\mathrm{D}}{\mathrm{D}t}(\bullet)=\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)(\bullet)+(\bullet)\,\boldsymbol{W}-\boldsymbol{W}(\bullet)-a[(\bullet)\,\boldsymbol{D}+\boldsymbol{D}(\bullet)]. \qquad (1.2b)$$

For the special values $a=1,-1,0$, the rate (1.2b) is commonly referred to as upper convected, lower convected and corotational, respectively. Here, the tensors $\boldsymbol{D}$ and $\boldsymbol{W}$ are the symmetric and skew parts of the velocity gradient,

$$\boldsymbol{D}=\tfrac{1}{2}(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^{\mathrm{T}}), \qquad (1.2c)$$

$$\boldsymbol{W}=\tfrac{1}{2}(\nabla\boldsymbol{v}-\nabla\boldsymbol{v}^{\mathrm{T}}), \qquad (1.2d)$$

with $(\nabla\boldsymbol{v})_{ij}=v_{i,j}\equiv(\partial/\partial x_j)\,(v_i)$ in Cartesian coordinates. The constants η, λ_1 and λ_2 are, respectively, the zero strain rate viscosity, relaxation time, and retardation time of the Maxwell–Jeffreys fluid. For $\lambda_1=\lambda_2=0$, (1.2a) reduces to the Newtonian constitutive law where $\hat{\boldsymbol{T}}$ is prescribed by gradients of the velocity field. Also, the Maxwell–Jeffreys model with upper convected rate is commonly called the Oldroyd fluid B. With these preliminaries, we return now to the central theme of this paper.

2. Exact closure models

The assumed velocity profile that reveals a separation of variables and the choice of velocity modal variables in this theory begins with a generalization of the von Kármán velocity ansatz (1921):

$$\boldsymbol{v}=[x\zeta_1(z,t)-y\psi(z,t)]\,\boldsymbol{e}_1+[y\zeta_2(z,t)+x\psi(z,t)]\,\boldsymbol{e}_2+v(z,t)\,\boldsymbol{e}_3. \qquad (2.1)$$

This is the most general linear polynomial in x and y which has reflection symmetry with respect to the (x,z)- and (y,z)-planes†; the original von Kármán ansatz assumes

† The truncated expansion (2.1) is presented for the purpose of exhibiting *exact* reductions of the three-dimensional problem. For the asymptotic scaling of §6, the velocity is assumed only to be expressible as a series in transverse coordinates x, y, which agrees with (2.1) to linear terms in the expansion.

axisymmetry, i.e. symmetry about all planes containing the z-axis, so that $\zeta_1 \equiv \zeta_2$. Here x, y, z denote the usual Cartesian coordinates, $\boldsymbol{e}_j$ $(j = 1, 2, 3)$ denote the corresponding base vectors and $\boldsymbol{e}_3$ represents the axial direction. Consistent with the assumption (2.1), we take the gravitational body force $\rho \boldsymbol{g}$ to be along $\boldsymbol{e}_3$.

Now, we evaluate momentum balance and incompressibility equations (1.1) for the velocity ansatz (2.1), which yields:

$$x\rho(\zeta_{1,t}+v\zeta_{1,z}+\zeta_1^2-\psi^2)-y\rho\{\psi_{,t}+v\psi_{,z}+\psi(\zeta_1+\zeta_2)\} = -p_{,x}+\hat{T}_{11,x}+\hat{T}_{12,y}+\hat{T}_{13,z}, \tag{2.2a}$$

$$x\rho\{\psi_{,t}+v\psi_{,z}+\psi(\zeta_1+\zeta_2)\}+y\rho(\zeta_{2,t}+v\zeta_{2,z}+\zeta_2^2-\psi^2) = -p_{,y}+\hat{T}_{12,x}+\hat{T}_{22,y}+\hat{T}_{32,z}, \tag{2.2b}$$

$$\rho(v_{,t}+vv_{,z}) = -p_{,z}+\rho g+\hat{T}_{13,x}+\hat{T}_{23,y}+\hat{T}_{33,z}, \tag{2.2c}$$

$$\zeta_1+\zeta_2+v_{,z} = 0. \tag{2.2d}$$

In this paper subscripts $,t$, $,z$, $,x$ and $,y$ denote partial differentiation with respect to time t, the axial coordinate z, and the transverse coordinates x and y, respectively.

In order to exhibit exact closure models, we need an ansatz for p and $\hat{T}_{ij}$ compatible with equation (2.1), so that equations (2.2) balance as polynomials in x and y. We now list two such cases in increasing order of complexity.

2.1. *Newtonian (Navier–Stokes) flows*

Let $\lambda_1 = 0 = \lambda_2, \eta \neq 0$, so that (1.2$a$) yields the Navier–Stokes constitutive assumption,

$$\hat{\boldsymbol{T}} = 2\eta \boldsymbol{D}. \tag{2.3}$$

The Newtonian constitutive law (2.3), together with the velocity ansatz (2.1), gives the stress components $\hat{T}_{ij}$ explicitly in terms of the functions ζ_1, ζ_2, ψ and v of z and t. Only $\hat{T}_{13}$ and $\hat{T}_{23}$ depend on x and y, and this dependence is linear. Equations (2.2a–c) imply that

$$p = p_0(z,t)+\tfrac{1}{2}x^2 p_1(t)+\tfrac{1}{2}y^2 p_2(t). \tag{2.4}$$

(Only p in equations (2.2a, b) depends implicitly on x and y, and the x and y dependence of the remaining terms is explicitly linear.) Hence in the Newtonian case, the velocity ansatz (2.1) implies that the pressure and stress components are also given by truncated power series in x and y, and equations (2.2) reduce to

$$\left.\begin{aligned}
\rho(\zeta_{1,t}+v\zeta_{1,z}+\zeta_1^2-\psi^2) &= \eta\zeta_{1,zz}-p_1(t),\\
\rho(\zeta_{2,t}+v\zeta_{2,z}+\zeta_2^2-\psi^2) &= \eta\zeta_{2,zz}-p_2(t),\\
\rho(\psi_{,t}+v\psi_{,z}+\psi(\zeta_1+\zeta_2)) &= \eta\psi_{,zz},\\
\rho(v_{,t}+vv_{,z}) &= -p_{0,z}+\rho g+\eta[(\zeta_1+\zeta_2)_{,z}+2v_{,zz}],\\
\zeta_1+\zeta_2+v_{,z} &= 0.
\end{aligned}\right\} \tag{2.5}$$

Equations (2.5) represent five equations for five unknowns, $(\zeta_1, \zeta_2, \psi, v, p_0)(z,t)$, with two arbitrary functions (p_1, p_2) of t.

2.2. *Upper convected Maxwell–Jeffreys, second-order exact truncation ansatz*

This exact closure model is significant in that the stress variables $\hat{T}_{ij}$ are *not* prescribed by the velocity field. When $\lambda_1 \neq 0$ in (1.2), the velocity ansatz (2.1) does not imply that the stress and pressure are also truncated power series in x and y.

Rather, in addition to (2.1) we posit a modal ansatz for $\hat{T}_{ij}$ and p that is consistent with the velocity ansatz (2.1), and which yields an exact, closed system of one-dimensional modal equations, derived from (1.1), (1.2). We demonstrate only the case $a = 1$. This viscoelastic closure model builds on an axisymmetric, Maxwell ($\lambda_2 = 0$) model due to Phan-Thien (1983).

With the velocity ansatz (2.1), we pose the following stress and pressure ansatz:

$$\left.\begin{aligned}
p(x,y,z,t) &= p_0(z,t)+p_1(t)\,x+p_2(t)\,y+p_3(t)\,x^2+p_4(t)\,y^2+p_5(t)\,xy,\\
\hat{T}_{11}(x,y,z,t) &= a_0(z,t)+a_1(z,t)\,x^2+a_2(z,t)\,y^2+a_3(z,t)\,xy+c_0(z,t)\,x+c_1(z,t)\,y,\\
\hat{T}_{21}(x,y,z,t) &= b_0(z,t)+a_4(z,t)\,x^2+a_5(z,t)\,y^2+a_6(z,t)\,xy+b_1(z,t)\,x+b_2(z,t)\,y,\\
\hat{T}_{31}(x,y,z,t) &= a_7(z,t)\,x+b_3(z,t)\,y,\\
\hat{T}_{32}(x,y,z,t) &= a_8(z,t)\,y+b_4(z,t)\,x,\\
\hat{T}_{22}(x,y,z,t) &= a_9(z,t)\,x^2+a_{10}(z,t)\,y^2+a_{11}(z,t)\,xy+a_{12}(z,t)+b_5(z,t)\,x+b_6(z,t)\,y,\\
\hat{T}_{33}(x,y,z,t) &= a_{13}(z,t).
\end{aligned}\right\} \quad (2.6)$$

Substituting (2.6) into the momentum conservation and incompressibility equations (2.2) (which already incorporates (2.1)), yields five modal equations for the modal variables, $(\zeta_1, \zeta_2, \psi, v, a_0, \ldots, a_{13}, c_0, c_1, b_0, \ldots, b_6, p_0)(z,t)$, and the time-dependent pressure variables $p_1(t), \ldots, p_5(t)$, and three constraint equations involving only the stress and pressure unknowns. Then we insert the velocity, pressure and stress ansatz (2.1), (2.6) into the upper convected ($a = 1$) Maxwell–Jeffreys constitutive law, and obtain 23 additional modal equations.

Thus, with the modal ansatz (2.1), (2.6), involving four velocity modal variables, one primitive pressure variable, five time-dependent pressure variables, and twenty-three stress variables, we obtain 28 equations in 28 unknowns $(\zeta_1, \zeta_2, \psi, v, p_0, a_0, \ldots, a_{13}, c_0, c_1, b_0, \ldots, b_{6,\,p_0})$ involving five arbitrary functions of t, $(p_1, \ldots, p_5)$, which are constrained by three conditions.

Remark. We do not explicitly present this exact closure model, nor do we consider the full class of solutions of these modal equations. The upshot here is simply that there exists an exact one-dimensional modal truncation for viscoelastic fluids, which fuels one's imagination that more general, approximate one-dimensional models exist.

These exact closure models also illustrate our next main point. That is, when a free surface is coupled to the fluid equations, exact closure is lost. As will be seen in the following section, the free-surface kinematic and kinetic boundary conditions effectively couple higher-order terms in the power series expansions for $\boldsymbol{v}$, p and $\hat{\boldsymbol{T}}$ to these lowest-order modal equations, yielding the classic closure difficulties.

3. Free-surface boundary conditions: compatibility with closure models

We are interested in modelling the dynamics of free jets. Here we describe the free surface, its modal variables, and the associated boundary conditions. We then describe the impact of a free surface on closure models and indicate how the free-surface boundary conditions influence the choice of stress modal variables. This point is expanded upon in Bechtel & Forest (1988).

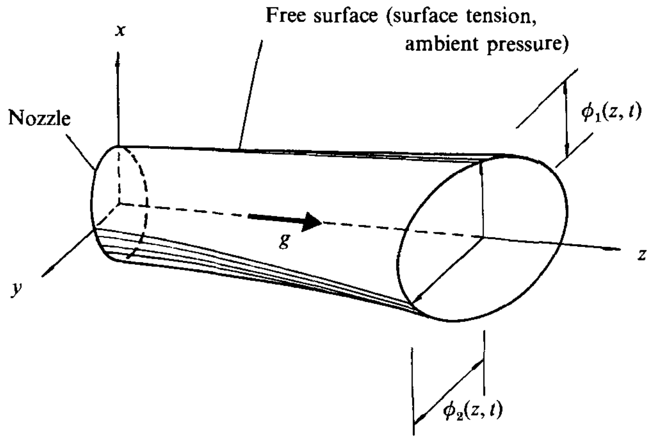

FIGURE 1. Free jet with elliptical cross-section.

The free surface, with assumed elliptical cross-section, is represented by

$$\frac{x^2}{\phi_1^2(z,t)}+\frac{y^2}{\phi_2^2(z,t)} = 1. \tag{3.1}$$

Each cross-section $z = z_0$ is an ellipse with semi-axis lengths ϕ_1, ϕ_2, which deform in z and t. See figure 1. The surface unknowns $\phi_1(z,t)$, $\phi_2(z,t)$ are additional modal variables. To complete the three-dimensional viscoelastic free-surface boundary-value problem, we adjoin to (1.1), (1.2) the interfacial boundary conditions (equations (3.2)–(3.6) below):

The kinematic boundary conditions. The free surface is convected with the fluid. From the velocity ansatz (2.1) and free surface ansatz (3.1), this condition yields:

$$\phi_{\alpha,t}+v\phi_{\alpha,z} = \phi_\alpha \zeta_\alpha \quad (\alpha = 1,2), \tag{3.2a}$$

$$(\phi_1^2-\phi_2^2)\,\psi = 0. \tag{3.2b}$$

The second condition is very restrictive: either there is no swirl ($\psi = 0$), or the swirling flow must be axisymmetric ($\phi_1 = \phi_2$ and $\zeta_1 = \zeta_2$). For the remainder of this paper, with the exception of this section on exact closure, we restrict to the case of no swirl, $\psi \equiv 0$. The alternative of axisymmetric swirl is pursued in Bechtel, Bolinger & Forest (1988). This restriction (3.2*b*) could be removed in two ways: (i) by allowing the cross-section (3.1) to rotate (see Caulk & Naghdi 1979*a* in the context of director theory; or, (ii) by allowing a higher-order flow geometry than (2.1). These generalizations are pursued elsewhere in Bechtel, Forest & Lin (1988*b*).

The kinetic boundary conditions. Shear stresses are assumed continuous across the fluid/ambient interface, whereas the normal stress is discontinuous. The jump in normal stress across the free surface is assumed to be balanced by the constant surface tension σ times the free surface mean curvature κ. These conditions state:

$$\boldsymbol{t}_{\mathrm{f}}-\boldsymbol{t}_{\mathrm{a}} = -\sigma\kappa\boldsymbol{n}, \tag{3.3a}$$

where $\boldsymbol{t}_{\mathrm{f}}$ and $\boldsymbol{t}_{\mathrm{a}}$ are the boundary stress vectors in the jet and ambient material, respectively, and $\boldsymbol{n}$ is the unit outward normal to the interface. We further assume the ambient material exerts a constant pressure p_{a}:

$$\boldsymbol{t}_{\mathrm{a}} = -p_{\mathrm{a}}\boldsymbol{n}. \tag{3.3b}$$

(Many physical applications require a more complicated ambient fluid model than (3.3*b*). Joseph *et al.* (1983), for example, incorporate the dynamics of the ambient fluid in order to study entrainment. We are currently extending the analysis to allow for the pursuit of these applications.)

For the elliptical free-surface ansatz (3.1), with the arbitrary stress (1.1*b*), the normal $\boldsymbol{n}$ and free-surface mean curvature κ are computed, and (3.3) becomes:

$$-(\sigma\kappa+p_{\mathrm{a}}-p|_{\partial})\phi_2\cos\theta = \hat{T}_{11}|_{\partial}\phi_2\cos\theta-\hat{T}_{13}|_{\partial} \times(\phi_1\phi_{2,z}\sin^2\theta+\phi_2\phi_{1,z}\cos^2\theta)+\hat{T}_{12}|_{\partial}\phi_1\sin\theta, \tag{3.4a}$$

$$-(\sigma\kappa+p_{\mathrm{a}}-p|_{\partial})\phi_1\sin\theta = \hat{T}_{22}|_{\partial}\phi_1\sin\theta-\hat{T}_{23}|_{\partial} \times(\phi_1\phi_{2,z}\sin^2\theta+\phi_2\phi_{1,z}\cos^2\theta)+\hat{T}_{12}|_{\partial}\phi_2\cos\theta, \tag{3.4b}$$

$$(\sigma\kappa+p_{\mathrm{a}}-p|_{\partial})(\phi_1\phi_{2,z}\sin^2\theta+\phi_2\phi_{1,z}\cos^2\theta) = \hat{T}_{13}|_{\partial}\phi_2\cos\theta +\hat{T}_{23}|_{\partial}\phi_1\sin\theta-\hat{T}_{33}|_{\partial}(\phi_1\phi_{2,z}\sin^2\theta+\phi_2\phi_{1,z}\cos^2\theta), \tag{3.4c}$$

where the symbol $|_{\partial}$ indicates evaluation on the boundary of the elliptical cross-section at

$$x = \phi_1\cos\theta, \quad y = \phi_2\sin\theta, \tag{3.5}$$

and the mean curvature κ is given by

$$\begin{aligned}\kappa(\theta,z,t) = &-[(\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta)(\phi_{1,zz}\phi_2\cos^2\theta+\phi_{2,zz}\phi_1\sin^2\theta)\\ &+2(\phi_1\phi_{2,z}-\phi_2\phi_{1,z})(\phi_1\phi_{1,z}-\phi_2\phi_{2,z})\cos^2\theta\sin^2\theta\\ &-\phi_1\phi_2(\phi_{1,z}^2\cos^2\theta+\phi_{2,z}^2\sin^2\theta+1)]\times[(\phi_1\phi_{2,z}\sin^2\theta+\phi_2\phi_{1,z}\cos^2\theta)^2\\ &+\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta]^{-\frac{3}{2}}.\end{aligned} \tag{3.6}$$

Given these free-surface boundary conditions (3.2)–(3.6), any closure model for free jets derived from the three-dimensional theory which is based on the elliptic von Kármán ansatz (2.1) (thus far, all are based on the axisymmetric special case) must respect these boundary conditions. This point is not completely checked in some models. Posited one-dimensional models for fluid jets (cf. Caulk & Naghdi 1979*a*, *b*, 1987; Green 1975, 1976; Bechtel *et al.* 1986) choose to demand that the kinematic conditions (3.2) be satisfied, but do not require the pointwise kinetic condition (3.4). We now investigate these free-surface constraints on the exact closure models of the previous section. Then, in §4, we return to the general situation when there is no exact power series truncation (those listed here are, to our knowledge, the only known exact truncations), and reassess the choice of modal variables.

3.1. *Free Surface Newtonian flows*

With the von Kármán velocity profile (2.1), Newtonian constitutive assumption (2.3) on $\hat{\boldsymbol{T}}$, and the resulting truncated power series representation (2.4) on p, the kinetic boundary conditions imply (when $\sigma \neq 0$) that the free jet is axisymmetric, i.e.

$$\zeta_1 = \zeta_2 = \zeta, \quad \phi_1 = \phi_2 = \phi, \quad p_1 = p_2. \tag{3.7}$$

This constraint eliminates the possibility of employing the exact one-dimensional closure model to investigate non-axisymmetric free jet behaviour. In addition, the kinetic boundary condition (3.4*b*) demands

$$\phi^2 \phi_{,z} \psi_{,z} = 0. \tag{3.8}$$

Thus, either the free surface is cylindrical, $\phi \equiv \phi_0(t)$, independent of z, or there is no swirl, $\psi \equiv 0$. We elect to satisfy (3.4*b*) by restricting $\psi \equiv 0$. Then the remaining kinetic boundary conditions (3.4*a*, *c*) impose the constraints:

$$p_0(z,t) = p_a - \tfrac{1}{2}\phi^2 p_1(t) + \sigma\kappa + \eta(2\zeta - \phi\phi_{,z}\zeta_{,z}), \tag{3.9}$$
$$\phi_{,z}(\phi\phi_{,z}\zeta_{,z} - 2\zeta) = \phi\zeta_{,z} - 2v_{,z}\phi_{,z}.$$

Thus, the primitive pressure variable $p_0(z,t)$ is prescribed and the second condition is a non-trivial constraint on ϕ, ζ and v.

Summarizing, we have three modal equations (2.5*a*, *d*, *e*), p_0 is prescribed by (3.9*a*), plus we have the constraint (3.9*b*). This free-surface Newtonian model now has three unknowns, (ζ, v, ϕ), with one arbitrary function $p_1(t)$, which must satisfy four independent equations. *The system is overdetermined.* The limited class of solutions excludes much axisymmetric jet behaviour of physical interest (see Bechtel & Forest 1988).

We close this example by remarking that the overdeterminism in this free-surface model arises because the free-surface boundary conditions (3.4) directly involve the stress and pressure modal variables, as defined by power series expansions in x and y.

3.2. *Free surface upper convected Maxwell–Jeffreys flows*

Without burdening the exposition with too many details, once again the kinetic boundary conditions yield an overdetermined system of modal equations. This is because the power series definition of stress and pressure modal variables, (2.6), forces the stress and pressure modal variables into the boundary conditions. Therefore, this exact closure model collapses when a free surface is adjoined.

4. Integrated momentum equations and free-surface boundary conditions: selection of stress modal variables

The upshot of §3 is the illustration of how the use of power series expansions for stress and pressure in x and y, in which the modal variables (or unknowns) are defined as the coefficients, can be a double-edged sword. More detailed information on the velocity field and stress is obtained in the jet cross-section by keeping higher powers in the expansion, but, with a free surface and the attendant boundary conditions, new constraints are introduced that severely restrict the exact one-dimensional jet models.

To derive one-dimensional jet models from the three-dimensional theory with the necessary flexibility to describe interesting behaviour, such as non-axisymmetric free jets and jets with swell, we retain the power series assumption (2.1) on $\boldsymbol{v}$, but choose stress and pressure unknowns to be integrals over the jet cross-section. This approach is taken by, among others, Matovich & Pearson (1969) and Denn *et al.* (1975) in the application to viscoelastic free jets, and by Green *et al.* (1974*a*) in the application to elastic rods.

This leads us to two important points. First, our power series ansatz for $\boldsymbol{v}$ limits the ability of this theory to meet velocity boundary conditions, such as no slip. Since boundary values of velocity are explicit combinations of the velocity modal variables (i.e. the coefficients in the power series expansion) and free-surface modal variables ϕ_1 and ϕ_2, the imposition of a condition on velocity at the boundary would constrain the velocity within the cross-section. However, our second point is that, historically, the reason for the use of area-averaged stress and pressure variables (rather than pointwise, power series expansions) is precisely to not limit the ability to meet the stress boundary conditions for free jets. As we have demonstrated in §3, with the power series expressions for stress and pressure there is not enough flexibility to meet the stress boundary conditions, for the same reason the $\boldsymbol{v}$ ansatz fails to meet flow boundary conditions. The boundary values of stress and pressure are explicit combinations of the stress and pressure modal variables (i.e. the coefficients in the power series expansions) and free-surface modal variables ϕ_1 and ϕ_2. Therefore the boundary conditions (3.2) and (3.4) are coupled to the model equations as severe constraints on the class of solutions of the modal equations, and hence limit the ability to model interesting free-jet phenomena (this point is expanded on in Bechtel & Forest 1988).

Thus, the bargain is made to give up detailed pointwise stress information in exchange for averaged and moment-averaged stress information and the flexibility to respect the interfacial boundary conditions. Area-averaged stress and pressure are chosen as the stress and pressure modal variables, and the equations relating them to the velocity and free-surface modal variables are obtained from the three-dimensional momentum conservation equations (2.2*a*–*c*) by integration over the jet cross-section. In the following we adopt this approach, and carefully indicate how free-surface boundary conditions are incorporated into the model, but still remain as a connection to the full three-dimensional boundary value problem (this last aspect of one-dimensional closure models is often neglected).

The first step is to compute certain cross-sectional area integrations and moment integrations of the conservation of momentum equations (2.2), which have been evaluated on the velocity ansatz (2.1). We make no *a priori* stress and pressure model ansatz. Recalling the discussion of the elliptical free surface (3.1), and the boundary conditions (3.2), (3.4), we hereafter restrict to no swirl, $\psi = 0$.

We compute the following integrations over the area A bounded by the ellipse, at fixed z, given by (3.1):

$$\iint_A \{x\cdot(2.2a)\}\,\mathrm{d}A, \quad \iint_A \{y\cdot(2.2b)\}\,\mathrm{d}A, \quad \iint_A (2.2c)\,\mathrm{d}A, \tag{4.1}$$

In a calculation which one must experience to appreciate, one uses the divergence theorem, integration by parts, and Leibniz' rule for differentiation of integrals, and all boundary terms from (4.1) involving p and $\hat{T}_{ij}$ either cancel, or what remains is precisely the linear combination that appears in the interfacial kinetic boundary conditions (3.4). Thus, one 'incorporates' the boundary conditions (3.4) into the integrated momentum equations; in other words, one replaces the boundary values of stress and pressure by the mean curvature, surface tension, and ambient pressure variables. The calculations are tedious, but straightforward, and are omitted. The result of these calculations is:

$$A_{131,z} - A_{11} = \sigma\phi_1\phi_2\int_0^{2\pi} \kappa\cos^2\theta\,\mathrm{d}\theta - \bar{p} + \tfrac{1}{4}\pi\rho\phi_1^3\phi_2(\zeta_{1,t} + v\zeta_{1,z} + \zeta_1^2), \tag{4.2a}$$

$$A_{232,z}-A_{22}=\sigma\phi_1\phi_2\int_0^{2\pi}\kappa\sin^2\theta\,\mathrm{d}\theta-\bar{p}+\tfrac{1}{4}\pi\rho\phi_2^3\phi_1(\zeta_{2,t}+v\zeta_{2,z}+\zeta_2^2), \tag{4.2b}$$

$$A_{33,z}=\bar{p}_{,z}-\pi\rho g\phi_1\phi_2+\pi\rho\phi_1\phi_2(v_{,t}+vv_{,z})$$
$$-\sigma\Big(\phi_1\phi_{2,z}\int_0^{2\pi}\kappa\sin^2\theta\,\mathrm{d}\theta+\phi_2\phi_{1,z}\int_0^{2\pi}\kappa\cos^2\theta\,\mathrm{d}\theta\Big), \tag{4.2c}$$

where κ is the mean curvature given by (3.4*e*). We append the incompressibility condition (2.2*d*), and the kinematic boundary conditions (3.2*a*),

$$v_{,z}+\zeta_1+\zeta_2=0,$$
$$\phi_{\alpha,t}+v\phi_{\alpha,z}=\phi_\alpha\zeta_\alpha\quad(\alpha=1,2). \tag{4.3}$$

The notation introduced above is for the following integrated stress and pressure variables:

$$A_{ij}\equiv\iint_A\hat{T}_{ij}\,\mathrm{d}A,\quad A_{ij1}\equiv\iint_A x\hat{T}_{ij}\,\mathrm{d}A,\quad A_{ij2}\equiv\iint_A y\hat{T}_{ij}\,\mathrm{d}A,$$
$$A_{ij12}\equiv\iint_A xy\hat{T}_{ij}\,\mathrm{d}A,\quad\text{etc.} \tag{4.4}$$
$$\bar{p}\equiv\iint_A(p-p_a)\,\mathrm{d}A.$$

We emphasize the above integrated equations (4.2) are *exact*, as are the equations (4.3). At this junction, the integrated stress and pressure variables, A_{11}, A_{22}, A_{33}, A_{131}, A_{232} and $\bar{p}$ are prime candidates for modal variables in our desired closure model. These are clearly natural choices from the conservation of momentum equations; it is precisely these averages and moment averages that allow the incorporation of the interfacial boundary conditions (3.4).

In the next two sections we produce the desired closure model, employing these integrated stress and pressure variables. First, in §5 we integrate the constitutive law (1.2) over the jet cross-section to obtain dynamical equations for A_{11}, A_{22}, A_{33}, A_{131}, A_{232}, and then in §6 we derive a self-consistent, *closed set* of modal equations for these stress variables and the velocity, pressure and free-surface modal variables $v, \zeta_1, \zeta_2, \bar{p}, \phi_1, \phi_2$.

Before we proceed, we return to the fate of the kinetic boundary conditions (3.4). At this point, it has been customary in the history of one-dimensional jet models to forget these boundary conditions, since, after all, they have been 'incorporated' into the integrated momentum equations (4.2). However, the equations (4.2) *do not imply* (3.4). Therefore, consistency with the full three-dimensional boundary-value problem of any one-dimensional model derived in this way demands that these boundary conditions can be met by the solutions of the one-dimensional modal equations.

It is at this point that the choice of stress and pressure modal variables, integrated *vs.* power series expansion in x and y, makes a dramatic difference. Recall that with the power series definition of stress and pressure modal variables, the stress boundary conditions (3.4) severely restrict the closure models. With the integrated choice this is not the case.

In effect, the integration technique to arrive at (4.2) has decoupled the boundary value unknowns $p|_\partial$, $\hat{T}_{ij}|_\partial$ from the principal modal variables v, ζ_1, ζ_2, ϕ_1, ϕ_2, $\bar{p}$, A_{11}, A_{22}, A_{33}, A_{131}, A_{232}. Information on the boundary-value unknowns $p|_\partial$, $\hat{T}_{ij}|_\partial$ can be obtained *a posteriori* from the free-surface stress boundary conditions (3.4) and the solution of the modal equations. Since for the Maxwell–Jeffreys model the boundary-value unknowns are independent of the principal modal variables, the stress boundary conditions (3.4) do not constitute constraints on the modal variables, as they did in the previous approach involving power series expansions for stress and pressure. This is the crucial advantage of using the integrated stress and pressure modal variables, as opposed to the coefficients in power series expansions.

This advantage is lost in the inviscid and Newtonian cases. For these degenerate cases the velocity ansatz (2.1) demands that stress and pressure are also truncated power series in x and y (see §2), and it can be shown that the closure models obtained by the two approaches, i.e. balancing polynomials in x and y or integrating over the jet cross-section, are equivalent and equally over-determined. In the inviscid and Newtonian special cases, the boundary-value unknowns are not independent of the principal model variables, so that the stress boundary conditions (3.4) are unavoidably constraints on the modal variables. See Bechtel & Forest (1988).

5. Integrated constitutive laws

Given these remarks, we now compute the area and area moment integrations of the stress equations (1.2) as dictated by the integrated momentum equations. Once again, in a calculation involving only integration by parts and Leibniz' rule, all of the boundary terms cancel in each of these integrated equations. Thus, we have the good fortune of boundary values of stress not entering into the stress resultant equations. The fact reflects the compatibility of the Maxwell–Jeffreys law with the elliptical von Kármán ansatz and the elliptical free surface. To get the A_{11} equation, we compute the area integral of the (1, 1) component of the matrix equations (1.2), $\iint_A (1.2)_{11}\,\mathrm{d}A$. Likewise, to get the equations for A_{22}, A_{33}, A_{232}, A_{131} we compute $\iint_A (1.2)_{22}\,\mathrm{d}A$, $\iint_A (1.2)_{33}\,\mathrm{d}A$, $\iint_A y(1.2)_{23}\,\mathrm{d}A$, $\iint_A x(1.2)_{13}\,\mathrm{d}A$, respectively. The *exact* results are:

$$A_{11}+\lambda_1[A_{11,t}+vA_{11,z}-((2a+1)\zeta_1+\zeta_2)A_{11}-(a+1)\zeta_{1,z}A_{131}]$$
$$= 2\pi\eta\phi_1\phi_2[\zeta_1+\lambda_2\{\zeta_{1,t}+v\zeta_{1,z}-2a\zeta_1^2-\tfrac{1}{8}(a+1)\phi_1^2\zeta_{1,z}^2\}], \quad (5.1a)$$

$$A_{22}+\lambda_1[A_{22,t}+vA_{22,z}-((2a+1)\zeta_2+\zeta_1)A_{22}-(a+1)\zeta_{2,z}A_{232}]$$
$$= 2\pi\eta\phi_1\phi_2[\zeta_2+\lambda_2\{\zeta_{2,t}+v\zeta_{2,z}-2a\zeta_2^2-\tfrac{1}{8}(a+1)\phi_2^2\zeta_{2,z}^2\}], \quad (5.1b)$$

$$A_{33}+\lambda_1[A_{33,t}+vA_{33,z}-(\zeta_1+\zeta_2+2av_{,z})A_{33}+(1-a)(\zeta_{1,z}A_{131}+\zeta_{2,z}A_{232})]$$
$$= 2\pi\eta\phi_1\phi_2[v_{,z}+\lambda_2\{v_{,zt}+vv_{,zz}-2av_{,z}^2+\tfrac{1}{8}(1-a)(\phi_1^2\zeta_{1,z}^2+\phi_2^2\zeta_{2,z}^2)\}], \quad (5.1c)$$

$$A_{131}+\lambda_1[A_{131,t}+vA_{131,z}-(2\zeta_1+\zeta_2+a(\zeta_1+v_{,z}))A_{131}$$
$$+\tfrac{1}{2}(1-a)(\zeta_{1,z}A_{1111}+\zeta_{2,z}A_{1212})-\tfrac{1}{2}(1+a)\zeta_{1,z}A_{3311}]$$
$$= \tfrac{1}{4}\pi\eta\phi_1^3\phi_2[\zeta_{1,z}+\lambda_2\{\zeta_{1,zt}+v\zeta_{1,zz}+\zeta_{1,z}(2(1-a)\zeta_1-(2a+1)v_{,z})\}], \quad (5.1d)$$

$$A_{232}+\lambda_1[A_{232,t}+vA_{232,z}-(\zeta_1+2\zeta_2+a(\zeta_2+v_{,z}))A_{232}$$
$$+\tfrac{1}{2}(1-a)(\zeta_{1,z}A_{1212}+\zeta_{2,z}A_{2222})-\tfrac{1}{2}(1+a)\zeta_{2,z}A_{3322}]$$
$$= \tfrac{1}{4}\pi\eta\phi_1\phi_2^3[\zeta_{2,z}+\lambda_2\{\zeta_{2,zt}+v\zeta_{2,zz}+\zeta_{2,z}(2(1-a)\zeta_2-(2a+1)v_{,z})\}]. \quad (5.1e)$$

These exact equations for A_{11}, A_{22}, A_{33}, A_{131}, A_{232} couple additional stress resultants, A_{1111}, A_{1212}, A_{3311}, A_{2222}, A_{3322}. We are thus led to the classic closure difficulty, where next we seek equations for these second moment area averages, which couples new stress resultants, and so on. As expected, there is no exact closure.

(We note that the closure difficulty exists only if $\lambda_1 \neq 0$ in the Maxwell–Jeffreys constitutive model. If $\lambda_1 = 0$, i.e. for the special cases of an inviscid fluid ($\eta = \lambda_1 = \lambda_2 = 0$), Newtonian fluid ($\lambda_1 = \lambda_2 = 0$) and second-order fluid ($\lambda_1 = 0$), the one-dimensional model (5.1) is closed; however, as discussed in the previous section, the model is overconstrained in these degenerate cases by the kinetic free-surface boundary condition (3.4), so that only very limited classes of solutions exist. We comment in passing that the same asymptotic analysis which will be found in the following sections to produce closure in the general case of $\lambda_1 \neq 0$ also relieves the overdeterminism of the degenerate cases with $\lambda_1 = 0$. A complete treatment can be found in Bechtel & Forest (1988).)

6. Asymptotic closure: slenderness scaling on the integrated momentum and constitutive equations

The next step is to restrict the exact equations (4.2), (5.1) to a 'slenderness' regime, by introducing a scaling analysis which is consistent with the elliptical von Kármán velocity ansatz (2.1), with $\psi = 0$. This scaling is modelled after that of Schultz & Davis (1982) in their study of axisymmetric Newtonian jets. First we non-dimensionalize the coordinates (x, y, z, t) and the modal velocity variables (ζ_1, ζ_2, v). Let r_0 = a typical lengthscale in the jet cross-section, L_a = a typical lengthscale in the axial direction, and t_0 = a typical timescale. The scaling hypothesis is:

$$x = \tilde{x} r_0, \quad y = \tilde{y} r_0, \quad z = \tilde{z} L_a, \quad t = \tilde{t} t_0, \tag{6.1a}$$

and the small parameter ϵ is the ratio of lengthscales,

$$\epsilon = \frac{r_0}{L_a} \ll 1. \tag{6.1b}$$

Thus, the approximation is that a typical radial scale is much shorter than a typical axial scale, and therefore is called the slenderness scaling.

The free surface and velocity modal variables are non-dimensionalized as

$$\phi_\alpha = \tilde{\phi}_\alpha r_0, \quad \zeta_\alpha = \tilde{\zeta}_\alpha \frac{1}{t_0}, \quad v = \tilde{v} v_a, \tag{6.1c}$$

where v_a is a characteristic axial velocity. To preserve the incompressibility condition (4.3*a*) and kinematic boundary conditions (4.3*b*) upon scaling, the characteristic velocity, length, and time scales must be related as

$$v_a = \frac{L_a}{t_0} = \frac{r_0}{\epsilon t_0}. \tag{6.1d}$$

Then
$$v^{(x)} = \epsilon v_a \tilde{x} \tilde{\zeta}_1 + O(\epsilon^3), \quad v^{(y)} = \epsilon v_a \tilde{y} \tilde{\zeta}_2 + O(\epsilon^3), \quad v^{(z)} = v_a \tilde{v} + O(\epsilon^2), \tag{6.1e}$$

so that the slenderness approximation in combination with the von Kármán velocity profile is equivalent to a slowly varying axial versus radial velocity ansatz.

[*Remark.* The scaled velocity formula (6.1*e*) includes higher-order corrections,

$O(\epsilon^2)$, to the von Kármán ansatz (2.1). These correspond to higher-order polynomial terms, $O(x^2, y^2, xy)$, in a general power series expansion for $\boldsymbol{v}$,

$$\begin{aligned}\boldsymbol{v} = &(a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3)\, \boldsymbol{e}_1 \\ &+ (b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3)\, \boldsymbol{e}_2 \\ &+ (c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3)\, \boldsymbol{e}_3 \\ &+ O(x^4, x^3 y, \ldots),\end{aligned} \tag{2.1$'$}$$

where a_i, b_i, c_i are functions of z and t. Symmetry considerations cause many of these coefficients to vanish.

Consistency demands that we return to the previous integrated momentum equations (4.2), incompressibility constraint and kinematic boundary conditions (4.3), and integrated constitutive law (5.1), and add corrections which result from the integration with the higher-order velocity expansion (2.1)$'$. These additional terms do not enter the lowest-order closure models of this paper, but yield higher-order corrections in the asymptotics. The precise form of these terms, along with the resolved question of consistency to higher order in the perturbation expansion, appear in Bechtel, Forest & Lin (1988).]

Next we scale the three-dimensional pressure and stress components as

$$p(x,y,z,t) = \tilde{p}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\frac{f}{r_0^2},$$

$$\hat{T}_{ij}(x,y,z,t) = \tilde{T}_{ij}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\frac{f}{r_0^2} \quad (i,j = 1,2,3), \tag{6.1f}$$

where f is a characteristic force scale. It then follows that the stress resultants scale as

$$A_{ij} = \iint_A \hat{T}_{ij}\,\mathrm{d}x\,\mathrm{d}y = \iint_A \frac{f}{r_0^2}\tilde{T}_{ij}\, r_0^2\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y} = f\tilde{A}_{ij}, \tag{6.1g}$$

where, for example,

$$\tilde{A}_{11}(\tilde{z},\tilde{t}) = \iint_A \tilde{T}_{11}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\,\mathrm{d}\tilde{x},\mathrm{d}\tilde{y}, \quad \tilde{A}: \frac{\tilde{x}^2}{\tilde{\phi}_1^2} + \frac{\tilde{y}^2}{\tilde{\phi}_2^2} = 1. \tag{6.1h}$$

Also, the moment resultants $A_{ij\alpha}$, $A_{ij\alpha\beta}$ and pressure resultant $\bar{p}$ scale as

$$\begin{aligned} A_{ij\alpha} &= \iint_A \hat{T}_{ij} x_\alpha\,\mathrm{d}x\,\mathrm{d}y = r_0 f\tilde{A}_{ij\alpha}, \\ A_{ij\alpha\beta} &= \iint_A \hat{T}_{ij} x_\alpha x_\beta\,\mathrm{d}x\,\mathrm{d}y = r_0^2 f\tilde{A}_{ij\alpha\beta}, \\ \bar{p} &= \iint_A (p - p_\mathrm{a})\,\mathrm{d}A = f\tilde{\bar{p}}, \end{aligned} \tag{6.1i}$$

where, for example, $$\tilde{A}_{232}(\tilde{z},\tilde{t}) = \iint_A \tilde{T}_{23}(\tilde{x},\tilde{y},\tilde{z},\tilde{t})\,\tilde{y}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}. \tag{6.1j}$$

It is important to note that the non-dimensionalized quantities $\tilde{\bar{p}}$, $\tilde{A}_{ij}$, $\tilde{A}_{ij\alpha}$, $\tilde{A}_{ij\alpha\beta}$ are $O(1)$.

Next we non-dimensionalize all of the equations (4.2), (5.1), keeping all terms for now, and collecting terms in powers of the slenderness ratio ϵ. By the usual abuse of notation, we drop all tildes on the non-dimensionalized coordinates and variables.

The resulting non-dimensionalized equations are:

$$-B(\epsilon A_{131,z}-A_{11}) = B\bar{p}+\frac{1}{W}\phi_1\phi_2[\chi_c^{(0)}+\epsilon^2\chi_c^{(1)}+\dots]-\tfrac{1}{4}\epsilon^2\phi_1^3\phi_2[\zeta_{1,t}+v\zeta_{1,z}+\zeta_1^2], \tag{6.2a}$$

$$-B(\epsilon A_{232,z}-A_{22}) = B\bar{p}+\frac{1}{W}\phi_1\phi_2[\chi_s^{(0)}+\epsilon^2\chi_s^{(1)}+\dots]-\tfrac{1}{4}\epsilon^2\phi_2^3\phi_1[\zeta_{2,t}+v\zeta_{2,z}+\zeta_2^2], \tag{6.2b}$$

$$B(A_{33,z}-\bar{p}_{,z}) = \frac{-\phi_1\phi_2}{F}+\frac{1}{W}[\phi_{1,z}\phi_2(\chi_c^{(0)}+\epsilon^2\chi_c^{(1)}+\dots)$$
$$+\phi_1\phi_{2,z}(\chi_s^{(0)}+\epsilon^2\chi_s^{(1)}+\dots)]+\phi_1\phi_2(v_{,t}+vv_{,z}), \tag{6.2c}$$

$$v_{,z}+\zeta_1+\zeta_2 = 0, \tag{6.2d}$$

$$\phi_{k,t}+v\phi_{k,z} = \phi_k\zeta_k \quad (k=1,2), \tag{6.2e,f}$$

$$A_{11}+A_1[A_{11,t}+vA_{11,z}-((2a+1)\zeta_1+\zeta_2)A_{11}-\epsilon(a+1)\zeta_{1,z}A_{131}]$$
$$= 2Z\phi_1\phi_2[\zeta_1+A_2\{\zeta_{1,t}+v\zeta_{1,z}-2a\zeta_1^2-\epsilon^2\tfrac{1}{8}(a+1)\phi_1^2\zeta_{1,z}^2\}], \tag{6.2g}$$

$$A_{22}+A_1[A_{22,t}+vA_{22,z}-((2a+1)\zeta_2+\zeta_1)A_{22}-\epsilon(a+1)\zeta_{2,z}A_{232}]$$
$$= 2Z\phi_1\phi_2[\zeta_2+A_2\{\zeta_{2,t}+v\zeta_{2,z}-2a\zeta_2^2-\epsilon^2\tfrac{1}{8}(a+1)\phi_2^2\zeta_{2,z}^2\}], \tag{6.2h}$$

$$A_{33}+A_1[A_{33,t}+vA_{33,z}-(\zeta_1+\zeta_2+2av_{,z})A_{33}-\epsilon(1-a)(\zeta_{1,z}A_{131}+\zeta_{2,z}A_{232})]$$
$$= 2Z\phi_1\phi_2[v_{,z}+A_2\{v_{,zt}+vv_{,zz}-2av_{,z}^2+\tfrac{1}{8}\epsilon^2(1-a)(\phi_1^2\zeta_{1,z}^2+\phi_2^2\zeta_{2,z}^2)\}], \tag{6.2i}$$

$$A_{131}+A_1[A_{131,t}+vA_{131,z}-(2\zeta_1+\zeta_2+a(\zeta_1+v_{,z}))A_{131}$$
$$+\tfrac{1}{2}\epsilon((1-a)(\zeta_{1,z}A_{1111}+\zeta_{2,z}A_{1212})-(1+a)\zeta_{1,z}A_{3311})]$$
$$= \tfrac{1}{4}\epsilon Z\phi_1^3\phi_2[\zeta_{1,z}+A_2\{\zeta_{1,zt}+v\zeta_{1,zz}+\zeta_{1,z}(2(1-a)\zeta_1-(2a+1)v_{,z})\}], \tag{6.2j}$$

$$A_{232}+A_1[A_{232,t}+vA_{232,z}-(2\zeta_2+\zeta_1+a(\zeta_2+v_{,z}))A_{232}$$
$$+\tfrac{1}{2}\epsilon((1-a)(\zeta_{2,z}A_{2222}+\zeta_{1,z}A_{1212})-(1+a)\zeta_{2,z}A_{3322}]$$
$$= \tfrac{1}{4}\epsilon Z\phi_2^3\phi_1[\zeta_{2,z}+A_2\{\zeta_{2,zt}+v\zeta_{2,zz}+\zeta_{2,z}(2(1-a)\zeta_2-(2a+1)v_{,z})\}], \tag{6.2k}$$

where
$$\chi_c^{(j)} = \frac{1}{\pi}\int_0^{2\pi}\kappa^{(j)}\cos^2\theta\,\mathrm{d}\theta, \quad \chi_s^{(j)} = \frac{1}{\pi}\int_0^{2\pi}\kappa^{(j)}\sin^2\theta\,\mathrm{d}\theta,$$
$$\kappa = \kappa^{(0)}+\epsilon^2\kappa^{(1)}+\dots,$$
$$\kappa^{(0)} = -\phi_1\phi_2(\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta)^{-\frac{3}{2}}, \tag{6.3}$$

$$\kappa^{(1)} = -\tfrac{3}{2}\phi_1\phi_2(\phi_1\phi_{2,z}\sin^2\theta+\phi_2\phi_{1,z}\cos^2\theta)^2(\phi_2^2\cos^2\theta+\phi_1^2\sin^2\theta)^{-\frac{5}{2}}$$
$$+(\phi_2^2\cos^2\theta+\phi_1^2\sin^2\theta)^{-\frac{3}{2}}\{\phi_1\phi_2(\phi_{1,z}^2\cos^2\theta+\phi_{2.z}^2\sin^2\theta)$$
$$-2\sin^2\theta\cos^2\theta(\phi_1\phi_{2,z}-\phi_2\phi_{1,z})(\phi_1\phi_{1,z}-\phi_2\phi_{2,z})\},$$

and $$B = \frac{ft_0^2}{\pi\rho r_0^2 L_a} = \frac{f}{\pi\rho r_0^2 v_a^2} = \frac{\text{viscoelastic and constraint pressure effects}}{\text{inertial effects}},$$

$$\frac{1}{F} = \frac{gt_0^2}{L_a} = \frac{\text{gravity effects}}{\text{inertial effects}},$$

$$\frac{1}{W} = \frac{\sigma t_0^2}{\rho r_0 L_a^2} = \frac{\sigma}{\rho r_0 v_a^2} = \frac{\text{surface tension (capillary) effects}}{\text{inertial effects}}, \tag{6.4}$$

$$Z = \frac{\pi\eta r_0^2}{t_0 f} = \frac{\pi\eta r_0^2 v_a}{L_a f},$$

$$\Lambda_1 = \frac{\lambda_1}{t_0}, \quad \Lambda_2 = \frac{\lambda_2}{t_0}.$$

The non-dimensional parameters F, W, $(BZ)^{-1}$ and Λ_1 are recognized as the Froude, Weber, Reynolds and Weissenberg numbers respectively. Z, Λ_1, Λ_2 are the non-dimensional zero strain rate viscosity, relaxation time and retardation time, respectively, of the fluid.

We emphasize these equations depict the balance among all the physical effects that are incorporated. (The higher-order velocity expansion $(2.1)'$ does not alter (6.2) to this order in ϵ.) In the slenderness approximation, $0 < \epsilon \ll 1$, these equations yield the ability to perform theoretical experiments in which the relative physical effects are adjusted through the non-dimensional parameters. To this end, $\{B, 1/W, 1/F, \Lambda_1, \Lambda_2, Z\}$, which measure the various properties of the free jet, are scaled in powers of the slenderness ratio ϵ,

$$B = B_0\epsilon^b, \quad \frac{1}{W} = \frac{1}{W_0}\epsilon^w, \quad \frac{1}{F} = \frac{1}{F_0}\epsilon^f, \quad \Lambda_j = \Lambda_{j0}\epsilon^{\lambda j}, \quad Z = Z_0\epsilon^c, \tag{6.5}$$

where $B_0, \ldots, Z_0$ are $O(1)$.

In a specific physical application, the exponents in (6.5) are dictated by the particular length- and timescales and material properties. However, from a theoretical perspective, this six-parameter family allows us the freedom to explore the consequences of a wide range of relative flow, geometrical and rheological properties.

We define the choice of integer exponents in (6.5) as the *regime* of free jet behaviour, as this choice reflects the relative magnitudes of competing physical effects.

To complete the asymptotics, we must also expand the non-dimensionalized dependent variables, v, ζ_1, ζ_2, ϕ_1, ϕ_2, $\bar{p}$, A_{ij}, $A_{ij\alpha}$, $A_{ij\alpha\beta}$:

$$\left.\begin{aligned}
v &= v^{(0)} + \epsilon v^{(1)} = \ldots,\\
\zeta_\alpha &= \zeta_\alpha^{(0)} + \epsilon\zeta_\alpha^{(1)} + \ldots \quad (\alpha = 1,2),\\
\phi_\alpha &= \phi_\alpha^{(0)} + \epsilon\phi_\alpha^{(1)} + \ldots \quad (\alpha = 1,2),\\
\bar{p} &= \bar{p}^{(0)} + \epsilon\bar{p}^{(1)} + \ldots,\\
A_{ij} &= A_{ij}^{(0)} + \epsilon A_{ij}^{(1)} + \ldots,\\
A_{ij\alpha} &= A_{ij\alpha}^{(0)} + \epsilon A_{ij\alpha}^{(1)} + \ldots \quad (i,j = 1,2,3; \quad \alpha = 1,2),\\
A_{ij\alpha\beta} &= A_{ij\alpha\beta}^{(0)} + \epsilon A_{ij\alpha\beta}^{(1)} + \ldots \quad (i,j = 1,2,3; \quad \alpha,\beta = 1,2).
\end{aligned}\right\} \tag{6.6}$$

In this paper we explicitly present only the lowest-order jet models in the slenderness limit (in another paper, Bechtel *et al.* (1988*b*), we exhibit the first-order corrections to the lowest-order equations, for the purpose of determining their effects on the lowest-order solutions). We therefore list the asymptotic equations retaining only the lowest power of ϵ within each physical term. Thus, specifying the physical properties will alter the relative order of the lowest terms we keep, but these will be the leading-order contributions from each physical property no matter which physical properties dominate. (All dependent variables in (6.7) are the leading-order contributions; we omit the superscript (0).)

$$B(A_{11}-\bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_c^{(0)} - \tfrac{1}{4}\epsilon^2\phi_1^3\phi_2(\zeta_{1,t}+v\zeta_{1,z}+\zeta_1^2), \tag{6.7a}$$

$$B(A_{22}-\bar{p} = \frac{1}{W}\phi_1\phi_2\chi_s^{(0)} - \tfrac{1}{4}\epsilon^2\phi_2^3\phi_1(\zeta_{2,t}+v\zeta_{2,z}+\zeta_2^2), \tag{6.7b}$$

$$B(A_{33,z}-\bar{p}_{,z}) = \frac{-1}{F}\phi_1\phi_2 + \frac{1}{W}(\phi_{1,z}\phi_2\chi_c^{(0)}+\phi_{2,z}\phi_1\chi_s^{(0)}) + \phi_1\phi_2(v_{,t}+vv_{,z}), \tag{6.7c}$$

$$v_{,z}+\zeta_1+\zeta_2 = 0, \tag{6.7d}$$

$$\phi_{\alpha,t}+v\phi_{\alpha,z} = \phi_\alpha\zeta_\alpha \quad (\alpha = 1,2), \tag{6.7e}$$

$$A_{11}+\Lambda_1[A_{11,t}+vA_{11,z}-((2a+1)\zeta_1+\zeta_2)A_{11}] = 2Z\phi_1\phi_2[\zeta_1+\Lambda_2\{\zeta_{1,t}+v\zeta_{1,z}-2a\zeta_1^2\}], \tag{6.7f}$$

$$A_{22}+\Lambda_1[A_{22,t}+vA_{22,z}-((2a+1)\zeta_2+\zeta_1)A_{22}] = 2Z\phi_1\phi_2[\zeta_2+\Lambda_2\zeta_{2,t}+v\zeta_{2,z}-2a\zeta_2^2\}], \tag{6.7g}$$

$$A_{33}+\Lambda_1[A_{33,t}+vA_{33,z}-(\zeta_1+\zeta_2+2av_{,z})A_{33}] = 2Z\phi_1\phi_2[v_{,z}+\Lambda_2\{v_{,zt}+vv_{,zz}-2av^2_{,z}\}], \tag{6.7h}$$

$$A_{131}+\Lambda_1[A_{131,t}+vA_{131,z}-(2\zeta_1+\zeta_2+a(\zeta_1+v_{,z}))A_{131}]$$
$$= \tfrac{1}{4}\epsilon Z\phi_1^3\phi_2[\zeta_{1,z}+\Lambda_2\{\zeta_{1,zt}+v\zeta_{1,zz}+\zeta_{1,z}(2(1-a)\zeta_1-(2a+1)v_{,z})\}], \tag{6.7i}$$

$$A_{232}+\Lambda_1[A_{232,t}+vA_{232,z}-(2\zeta_2+\zeta_1+a(\zeta_2+v_{,z}))A_{232}]$$
$$= \tfrac{1}{4}\epsilon Z\phi_2^3\phi_1[\zeta_{2,z}+\Lambda_2\{\zeta_{2,zt}+v\zeta_{2,zz}+\zeta_{2,z}(2(1-a)\zeta_2-(2a+1)v_{,z})\}], \tag{6.7j}$$

where
$$\chi_c^{(0)} = -\frac{\phi_1\phi_2}{\pi}\int_0^{2\pi}\frac{\cos^2\theta\,\mathrm{d}\theta}{(\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta)^{\frac{3}{2}}},$$

$$\chi_s^{(0)} = -\frac{\phi_1\phi_2}{\pi}\int_0^{2\pi}\frac{\sin^2\theta\,\mathrm{d}\theta}{(\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta)^{\frac{3}{2}}}.$$

This analysis reveals the asymptotic balance of the competing physical effects on viscoelastic slender free-jet behaviour. We are finally in a position to exhibit one-dimensional closure models. We specify a particular jet regime through a choice of integer exponents in (6.5) and obtain the lowest-order equations. There is clearly a tremendous amount of latitude in exploring all the specialized closure models which derive from our general construction.

As will be shown in the next section, existing one-dimensional theories correspond to the axisymmetric, steady forms of the lowest-order equations with certain physical effects suppressed to higher order. Before connecting with the existing models, however, we first illustrate with three more general (non-axisymmetric and time dependent) regimes.

As one example of a one-dimensional closure model for a particular jet regime, consider the case where all of the parameters in the set $\{B, 1/W, 1/F, A_1, A_2, Z\}$ are $O(\epsilon^0)$, i.e. all exponents in (6.5) are zero. The lowest-order equations in the asymptotic expansion are then

$$B(A_{11}-\bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_c^{(0)}, \quad B(A_{22}-\bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_s^{(0)}, \tag{6.8a, b}$$

$$B(A_{33,z}-\bar{p}_{,z}) = -\frac{1}{F}\phi_1\phi_2+\frac{1}{W}(\phi_{1,z}\phi_2\chi_c^{(0)}+\phi_{2,z}\phi_1\chi_s^{(0)})+\phi_1\phi_2(v_{,t}+vv_{,z}), \tag{6.8c}$$

$$v_{,z}+\zeta_1+\zeta_2 = 0, \quad \phi_{1,t}+v\phi_{1,z} = \phi_1\zeta_1, \quad \phi_{2,t}+v\phi_{2,z} = \phi_2\zeta_2, \tag{6.8d–f}$$

$$A_{11}+A_1[A_{11,t}+vA_{11,z}-((2a+1)\zeta_1+\zeta_2)A_{11}] = 2Z\phi_1\phi_2[\zeta_1+A_2(\zeta_{1,t}+v\zeta_{1,z}-2a\zeta_1^2)], \tag{6.8g}$$

$$A_{22}+A_1[A_{22,t}+vA_{22,z}-((2a+1)\zeta_2+\zeta_1)A_{22}] = 2Z\phi_1\phi_2[\zeta_2+A_2(\zeta_{2,t}+v\zeta_{2,z}-2a\zeta_2^2)], \tag{6.8h}$$

$$A_{33}+A_1[A_{33,t}+vA_{33,z}-(\zeta_1+\zeta_2+2av_{,z})A_{33}] = 2Z\phi_1\phi_2[v_{,z}+A_2(v_{,zt}+vv_{,zz}-2av_{,z}^2)]. \tag{6.8i}$$

In this regime, inertial effects, surface tension and gravity are all leading order in the axial direction (see equation (6.8c)). Note that this demands inertial effects to be higher order in the transverse directions (see equations (6.8a, b)). Viscosity, relaxation and retardation effects are all leading order in the constitutive model in this regime.

In this special regime, the lowest-order equations are a closed set of nine equations for the nine modal variables, $\phi_1^{(0)}$, $\phi_2^{(0)}$, $v^{(0)}$, $\zeta_1^{(0)}$, $\zeta_2^{(0)}$, $\bar{p}^{(0)}$, $A_{11}^{(0)}$, $A_{22}^{(0)}$, $A_{33}^{(0)}$. The shear stress resultants $A_{131}^{(0)}$, $A_{232}^{(0)}$ decouple to lowest order from equations (6.8), and appear in the problem for the first-order corrections $\phi_1^{(1)}$, $\phi_2^{(1)}$, $v^{(1)}$, etc. (see Bechtel *et al.* 1988b). The physical predictions of the closure model (6.8) for a variety of parameter values and steady nozzle conditions are explored in Bechtel *et al.* (1987, 1988a) and Bechtel, Forest & Hyman (1988).

As an example of a one-dimensional closure model for a different jet regime, consider the case where A_1 is $O(\epsilon^0)$, and B, $1/W$, $1/F$, A_2, Z are $O(\epsilon^2)$. For this regime the lowest-order equations in the asymptotic expansion are

$$B(A_{11}-\bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_c^{(0)}-\tfrac{1}{4}\phi_1^3\phi_2(\zeta_{1,t}+v\zeta_{1,z}+\zeta_1^2), \tag{6.9a}$$

$$B(A_{22}-\bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_s^{(0)}-\tfrac{1}{4}\phi_2^3\phi_1(\zeta_{2,t}+v\zeta_{2,z}+\zeta_2^2), \tag{6.9b}$$

$$0 = \phi_1\phi_2(v_{,t}+vv_{,z}), \tag{6.9c}$$

$$v_{,z}+\zeta_1+\zeta_2 = 0, \tag{6.9d}$$

$$\phi_{1,t}+v\phi_{1,z} = \phi_1\zeta_1, \quad \phi_{2,t}+v\phi_{2,z} = \phi_2\zeta_2, \tag{6.9e, f}$$

$$A_{11}+A_1[A_{11,t}+vA_{11,z}-((2a+1)\zeta_1+\zeta_2)A_{11}] = 0, \tag{6.9g}$$

$$A_{22}+A_1[A_{22,t}+vA_{22,z}-((2a+1)\zeta_2+\zeta_1)A_{22}] = 0. \tag{6.9h}$$

In this regime inertial effects are important for motion within the jet cross-section (see 6.9a, b), and gravity is neglected to leading order. These choices demand that momentum is conserved in the axial direction (equation (6.9*c*)). Only relaxation effects are included to leading order in the constitutive model, equations (6.9*g, h*).

In this regime, the lowest-order equations are a closed set of eight equations for the eight modal variables $\phi_1^{(0)}$, $\phi_2^{(0)}$, $v^{(0)}$, $\zeta_1^{(0)}$, $\zeta_2^{(0)}$, $\bar{p}^{(0)}$, $A_{11}^{(0)}$, $A_{22}^{(0)}$. The axial stress resultant $A_{33}^{(0)}$, and shear stress resultants $A_{131}^{(0)}$, $A_{232}^{(0)}$ decouple from this lowest-order problem.

The behaviour predicted by the closed equations (6.9) differs significantly from the behaviour predicted by equations (6.8), a reflection of the disparate parameter specifications. Again, we defer discussion of the predictions of this specific closure model to Bechtel *et al.* (1988*a*).

As a third example, consider the particular jet regime where B, $1/W$, $1/F$ are $O(\epsilon^3)$. Then, from (6.7), we obtain the lowest order equations:

$$\left.\begin{aligned} 0 &= \phi_1^3\phi_2(\zeta_{1,t}+v\zeta_{1,z}+\zeta_1^2),\\ 0 &= \phi_2^3\phi_1(\zeta_{2,t}+v\zeta_{2,z}+\zeta_2^2),\\ 0 &= \phi_1\phi_2(v_{,t}+vv_{,z}),\\ v_{,z}&+\zeta_1+\zeta_2 = 0,\\ \phi_{1,t}+v\phi_{1,z} &= \phi_1\zeta_1, \quad \phi_{2,t}+v\phi_{2,z} = \phi_2\zeta_2. \end{aligned}\right\} \tag{6.10}$$

In this regime, only inertial effects are leading order. Here the lowest-order equations are a set of *six* equations for the *five* unknowns $\phi_1^{(0)}$, $\phi_2^{(0)}$, $v^{(0)}$, $\zeta_1^{(0)}$, $\zeta_2^{(0)}$, which are easily shown to be overconstrained, and in fact incompatible. With this regime we have demonstrated another important result of this analysis: the ability to determine what properties of slender viscoelastic free jets combine to produce consistent one-dimensional closure, and which do not.

Many other specialized closure models are clearly available. We refer to Bechtel *et al.* (1987, 1988*a*) and Bechtel, Forest & Hyman (1988) for applications which have already derived from this work. Additional applications are planned.

7. Contact with existing one-dimensional theories

To illustrate the comprehensive nature of the above analysis, we now indicate how several widely referenced one-dimensional models for Newtonian and viscoelastic free jets are obtained by specification of particular jet regimes, and by reduction to the steady, axisymmetric forms. We list the order of magnitude of the parameters B, W, F, Z, Λ_1, Λ_2 in the slenderness ratio which produce exemplary existing models from our system (6.7) as the lowest-order equations.

The axisymmetric, steady form of the lowest-order equations with the parameters B, W, F, Z all $O(\epsilon^0)$ and the parameters Λ_1, Λ_2 both $O(\epsilon)$ is the Newtonian thin filament model, equation (34) from Matovich & Pearson (1969). In this regime, Newtonian viscosity, surface tension and gravity are leading order, with the elastic and second-order viscosity effects suppressed to higher order. (Recall the Reynolds number $R = (BZ)^{-1}$.)

The axisymmetric, steady form of the lowest order equations with the parameters B, W, F, Z, Λ_2 all specified as $O(\epsilon^0)$, Λ_1 specified as $O(\epsilon)$, and the rate parameter a taken as -1 (lower convected rate) is the second order, non-Newtonian thin filament model, equation (53) in Matovich & Pearson (1969).

The one-dimensional viscoelastic model in Tanner (1985) (equations (7.31)–(7.33)) is obtained as the axisymmetric, steady form of the lowest-order equations from (6.7) with the parameters B, Z, A_1 specified as $O(1/\epsilon)$, the parameters W, F, A_2 as $O(\epsilon^0)$, the rate parameter a taken as 1 (upper convected rate), and the choice of notation

$$A_{11} = P\phi^2, \quad A_{33} = T\phi^2.$$

Recalling the definitions

$$A_{11} \equiv \iint_{\text{cross-section}} \hat{T}_{11}\,\mathrm{d}A, \quad A_{22} \equiv \iint_{\text{cross-section}} \hat{T}_{22}\,\mathrm{d}A, \quad A_{33} \equiv \iint_{\text{cross-section}} \hat{T}_{33}\,\mathrm{d}A,$$

we see that $T(z)$ is the average normal stress over the jet cross-section in the axial direction $\boldsymbol{e}_3$, and $P(z)$ is the average normal stress in the transverse directions. In this regime the leading-order effects are viscosity and elasticity, with inertia in the axial direction, surface tension, gravity and retardation time effects suppressed. The axisymmetric, steady form of this regime with upper convected rates is also the model of Denn *et al.* (1975) (equations (12)–(14)), with the ratio ν of stress differences in their constitutive model taken as zero, and the model of Denn & Marrucci (1977), with a spectrum of one relaxation time. The axisymmetric, time-dependent form of this regime is the viscoelastic model, equations (10)–(13), in Fisher & Denn (1976) with the power law viscosity parameter n in their model set equal to 1.

The one-dimensional model in Gupta *et al.* (1986) for a free jet of an Oldroyd fluid B is obtained as the axisymmetric, steady form of the lowest-order equations from (6.7) with the parameters B, Z, A_1, A_2 all $O(1/\epsilon)$, the parameters W, F both $O(1)$, and the rate parameter a taken as 1 (upper convected rates).

The axisymmetric form of the leading-order equations from our system (6.7) with the parameters B, Z, A_1 taken as $O(1)$, the parameters $1/W$, $1/F$, A_2 taken as $O(\epsilon)$, and the rate parameter a chosen to be 1 (upper convected rate) is equivalent to the axisymmetric, time-dependent model, equations (2–1), of Beris & Liu (1988). In this regime the leading-order effects are viscosity, elasticity and inertia in the axial direction, with surface tension, gravity and retardation time effects suppressed.

8. Concluding remarks

We have satisfied the goals set in the §1. Beginning with the full three-dimensional viscoelastic free surface boundary-value problem, we have derived, by slenderness asymptotics, a comprehensive framework of one-dimensional closure models for slender, free viscoelastic jets. The physical effects of inertia, gravity, viscosity, elasticity, surface tension, curvature, and the free-surface boundary conditions involving surface tension, ambient pressure and the curvature of the free surface, are represented in the one-dimensional modal equations, and most importantly, these effects appear as they derive from the full three-dimensional free-surface boundary-value problem. These resultant one-dimensional equations have the flexibility to vary the relative strengths of the physical properties of the fluid and interface. Existing one-dimensional theories correspond to special cases within our general framework.

In future papers, we will examine the full steady and dynamical consequences of specific one-dimensional closure models. For example, how do these one-dimensional systems reflect three-dimensional jet instability mechanisms? Another follow-up to this work is the analysis of the next-order corrections to the lowest-order closure models in Bechtel *et al.* (1988*b*). These higher-order equations allow us to test the predictions of the lowest-order models, to determine if neglected effects become important, and to obtain more detailed information about the three-dimensional flow.

It remains an open and challenging problem as to whether this type of analysis can be brought to bear on development of reduced closure models in three-dimensional flows other than slender jets.

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